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Multifractal Structure of Fully Developed Hydrodynamic Turbulence. I. Kolmogorov's Third Hypothesis Revisited

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We discuss intermittency effects in fully developed hydrodynamic turbulence. It is shown that the application of the bounded log-normal distribution to the fluctuations of the local energy dissipation rate resolves some basic difficulties related to Kolmogorov's third hypothesis and gives a good agreement with experiment. The nonlinear interaction of the large-scale and inertial-range turbulent pulsations of the velocities may explain the observable characteristics of the intermittency. We give also a detailed comparison of the results obtained with the use of the bounded log-normal distribution with that obtained in the framework of the homogeneous and random β-models, a two-scale Cantor set approximation, and the original unbounded log-normal distribution suggested by Kolmogorov and Obukhov.

KEY WORDS: Intermittency effects; Kolmogorov's third hypothesis; bounded log-normal distribution; multifractal structure of turbulence.

1. INTRODUCTION

Kolmogorov's 1941 theory[1,2] suggested a universal scenario of random motion in high-Reynolds-number incompressible fluid turbulence. Its basic feature is a self-similar relay cascade of kinetic energy from the large (injection) scales to the small (dissipation) scales retaining the mean rate of energy dissipation. However, as remarked by Landau and Lifshitz,[2] the fluctuations in the local rate of energy transfer should also be taken into account. Through in the main approximation Kolmogorov's predictions appear to be in good correspondence with experiments,[1] subsequent data showed regular deviations from the Kolmogorov scaling laws for the

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weighted relative velocity averages (recent experimental results and further references can be found in refs. 3–6; a review is given ref. 7). These deviations are just related to the fluctuations in local energy transfer and are called usually intermittency effects.

Various particular mechanisms of intermittency have been discussed in numerous papers (the different stages of those investigations are partially summarized in refs. 1, 8, and 9). The most popular theories include: the log-normal model of fluctuations of the local energy transfer rate by Kolmogorov[10] and Obukhov,[11] the β-model[12] and its modifications,[13–16] and the two-scale Cantor set approximation for the energy dissipation distribution.[17,18] Since the rigorous derivation of the turbulence characteristics from first principles has not been achieved, the physical status of all these models remains mainly heuristic and needs further theoretical investigation and more detailed comparison with the experimental results.

We will show in this paper that the application of the bounded log-normal distribution for the fluctuations of energy dissipation rate instead of the original unbounded Kolmogorov–Obukhov distribution resolves some principle difficulties related to this model and gives good agreement with experiment. The mechanism based on the nonlinear interaction of the large-scale and inertial-range turbulent pulsations of the velocities may explain the main observable characteristics of intermittency. The results obtained in the framework of various theories are summarized and compared.

2. LOG-NORMAL MODEL

The experiments show (cf., e.g., ref. 6) that, strictly speaking, the intermittency characteristics may depend on whether the direct non-centered fluctuations of energy dissipation rate per unit mass ε are considered or the centered fluctuations near the mean value $\varepsilon - \bar{\varepsilon}$ are investigated. Below we will restrict ourselves to the first case and imply everywhere the fulfilment of inequalities $\ln(L/r) \gg r \gg l_\eta$, where L and l_η are the external and internal scales of turbulence, respectively.

According to Kolmogorov[10] and Obukhov,[11] the local fluctuations of energy transfer rate in a fluid volume with length scale $\sim r$ can be described with the use of a log-normal distribution:

$$P(\varepsilon_r) = \frac{1}{\bar{\varepsilon}\sigma_r(2\pi)^{1/2}} \exp\left\{\sigma_r^2 - \frac{[\ln(\varepsilon_r/\bar{\varepsilon}) + 3\sigma_r^2/2]^2}{2\sigma_r^2}\right\} \tag{1}$$

$$\sigma_r^2 = \mu \ln(L/r) \tag{2}$$

The moments $\langle \varepsilon_r^q \rangle$ are determined by

$$\langle \varepsilon_r^q \rangle = \int_0^\infty d\varepsilon_r \, \varepsilon_r^q P(\varepsilon_r) = \bar{\varepsilon}^q (L/r)^{\mu q(q-1)/2} \tag{3}$$

Equations (1)–(3) are frequently called Kolmogorov's third hypothesis. We have used the simplified expression for the variance σ_r^2 in Eq. (2) instead of

$$\sigma_r^2 = A + \mu \ln(L/r)$$

in the original version,[10,11] since the constant A can change only the numerical factors in Eq. (3) and its values are experimentally not known.

Equations (1)–(3) are the particular realization of the general model of random fragmentation considered by Kolmogorov.[19] In the context of turbulent cascade (see, e.g., refs. 10–22) it means that an energy flow transferred at the nth step of consecutive fragmentation ε_n (starting at scales $\sim L$) is connected to a previous value ε_{n-1} by the relationship

$$\varepsilon_n = a_n \varepsilon_{n-1}$$

$$\varepsilon_n = a_n a_{n-1} \cdots a_1 \varepsilon$$

where $\{a_n\}$ are random fractions ($0 < a_n < 1$). Taking the logarithm of ε_n, it is easy to see that $\ln \varepsilon_n$ is transformed into a sum of random variables, which is distributed under some general restrictions accordinig to the Gaussian law[19] (the more general situation has been discussed by Mandelbrot[23]).

The fluctuations of the relative velocities $\delta v_r = |\mathbf{v}(\mathbf{R}+\mathbf{r}) - \mathbf{v}(\mathbf{R})|$ are correspondingly determined by

$$\langle \delta v_r^q \rangle = \langle \varepsilon_r^{q/3} \rangle r^{q/3} \propto r^{q/3 - \mu_q/3} \tag{4}$$

In what follows v_r will always be understood in the sense of δv_r. Comparing Eqs. (4) and (3), one obtains

$$\mu_q = \frac{\mu}{2} q(q-1) \tag{5}$$

On the other hand, as has been proved by Novikov,[22] the corrections μ_q cannot increase to the limit of large q stronger than linear, in the contradiction to Eq. (5). Experimentally, the good correspondence with the log-normal model occurs for low values of q and the divergence from the log-normal predictions (1)–(5) is observed beginning approximately at $q \gtrsim 3$. We will show below how these difficulties can be resolved.

The necessary modification of the original Kolmogorov–Obukhov log-normal distribution is naturally related to the general physical picture of the Kolmogorov turbulence. The energy is injected at the large scales $\sim L$ and then is consecutively repumped to the dissipation range (at the scales $\sim l_\eta$). Thus, the inertial interval is bounded by the energy injection range (at $r \sim L$) and by the dissipation range (at $r \sim l_\eta$). Taking into account these two boundaries, it is evident that the limits of integration in Eq. (3) should be replaced by $\varepsilon_{\max,r}$ and $\varepsilon_{\min,r}$. If the saddle value of the expression under the integral in Eq. (3) is placed within the interval $(\varepsilon_{\min,r}, \varepsilon_{\max,r})$, then the influence of the boundaries can be neglected and the original unbounded Kolmogorov–Obukhov distribution is applicable. However, the saddle value $\varepsilon_{s,r}$ tends beyond the interval $(\varepsilon_{\min,r}, \varepsilon_{\max,r})$ with the increase of $|q|$ (where $|q|$ is the modulus of q) and the applicability of the unbounded distribution fails at large values of $|q|$. The logarithmic character of the relevant changes in ε (log-normality of distribution) is naturally important, since even in the most favorable case of atmospheric turbulence with enormous Reynolds numbers $\mathrm{Re} \sim 10^8$–10^9 the respective changes in logarithms remain only an order of magnitude. This is why $\bar{\varepsilon}_{s,r}$ grows beyond $(\varepsilon_{\min,r}, \varepsilon_{\max,r})$ rather rapidly even at relatively modest values of q (see below). Next, it is reasonable to assume that the corresponding characteristic values of the maximum and minimum energy dissipation rates $\varepsilon_{\max,r}$ and $\varepsilon_{\min,r}$ for a fluid volume with sizes $\sim r$ are also scaled according to

$$\varepsilon_{\max,r}/\bar{\varepsilon} \approx c_1 (L/r)^{\alpha_1} \tag{6}$$

$$\varepsilon_{\min,r}/\bar{\varepsilon} \approx c_2 (L/r)^{-\alpha_2} \tag{7}$$

where $\alpha_1 > 0$, $\alpha_2 > 0$. In the limit $\ln(L/r) \gg 1$ all moments $\langle \varepsilon_r^q \rangle$ will mainly be determined by the exponents α_1 and α_2, while the particular values of the constants c_1 and c_2 are not important within logarithmic accuracy. Figure 1 illustrates the systematic increase of the intermittency effects with a decrease of scale size.

Using the saddle value of the unbounded log-normal distribution (1)–(3) for a rough estimate, the criterion of applicability of the Kolmogorov–Obukhov intermittency approximation may be written in the form

$$-\alpha_2 > (q - 1/2)\mu > \alpha_1 \tag{8}$$

As can easily be proved (see also below), the quadratic corrections (5) will be replaced by linear ones with the increase of $|q|$ and beyond the interval (8) in accordance with Novikov's argumentation.[22] It is worth also noting

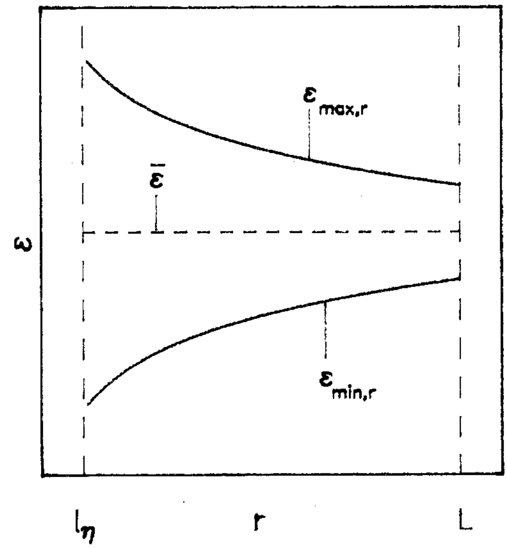

Fig. 1. Schematic increase of intermittency with a decrease of scale size.

the correspondence with ideas discussed in weak turbulence models, where the boundaries in the energy space also play an important role in the general balance of the energy transfer even for the local spectra.[24]

Numerical estimates of the constants α_1, α_2, and μ can be obtained from the following physical suggestions. In the first approximation of the purely Kolmogorov turbulence the nonlinear interactions of eddies with the same characteristic size $\sim r$ are the most important. If only these interactions are retained, then within the framework of such a truncation scheme the local energy transfer rate will approximately remain constant,

$$\varepsilon_r \sim v_r^2/\tau_r \sim v_r^3/r \sim \bar{\varepsilon} \tag{9}$$

At the next step the nonlinear coupling of the turbulent eddies with two drastically different lengths should be taken into account, while the simultaneous interaction of eddies with three different length scales $\sim L$, $\sim r$, andc $\sim l_\eta$ can still be neglected. Since experiments suggest that the intermittency corrections are expanded with respect to the large ratio (L/r) rather than (r/l_η), we will restrict ourselves to the coupling of macroscale energy injection pulsations with size $\sim L$ and inertial-range $(L \gg r \gg l_\eta)$ turbulent velocities. The perturbation series also suggest on the dominant role of the infrared (small-wavenumber) divergences over the ultraviolet (large-wavenumber) ones (see, e.g., refs. 25–28). The particular dynamical aspects of this interaction cannot be discussed within the framework of our qualitative considerations, but the possible scenario may be analogous to that suggested by Siggia[29] and supported later by computer simulations.[30] Very roughly one could say that an intermittent relative velocity with scale $\sim r$ now has the admixture of velocity pulsations with scale $\sim L$, i.e.,

$$\mathbf{v}_r^{(i)}(t) \approx a_r(t)\, \mathbf{v}_r(t) + a_L(t)\, \mathbf{v}_L(t)$$

with more or less random triggering of the coefficients $a_r(t)$ and $a_L(t)$ $(0 < \alpha < 1)$. Since the Kolmogorov picture has been used as the first approximation, this implies that the averaged admixture is not very strong,

$$\langle a_L(t)\, |\mathbf{v}_L(t)| \rangle < \langle a_r(t)\, |\mathbf{v}_r(t)| \rangle$$

The nonlinear coupling of the turbulent pulsations with length scales $\sim L$ and $\sim r$ causes fluctuations in local energy transfer rates ε_r and gives the following estimates:

$$\varepsilon_{\max,r} \sim v_L^2/\tau_r \sim v_L^2 v_r/r \tag{10}$$

$$\varepsilon_{\min,r} \sim v_r^2/\tau_L \sim v_r^2 v_L/L \tag{11}$$

In the large length scale limit $(r \sim L)$ the intermittency fluctuations are relatively weak (see (Fig. 1) and the corresponding random energy transfer rates ε_L are approximately self-averaging (i.e., $\langle \varepsilon_L^q \rangle \approx \bar{\varepsilon}_L^q$). Substituting in the first approximation the Kolmogorov value $v_r \sim \bar{\varepsilon}^{1/3} r^{1/3}$ into Eqs. (10) and (11), one obtains immediately

$$\varepsilon_{\max,r}/\bar{\varepsilon} \sim (L/r)^{2/3} \tag{12}$$

$$\varepsilon_{\min,r}/\bar{\varepsilon} \sim (L/r)^{-2/3} \tag{13}$$

i.e., $\alpha_1 \approx \alpha_2 \approx 2/3$.

A similar (though more arbitrary) argument can be applied to the variance constant μ. Replacing in $\varepsilon_r^2 \sim v_r^6/r^2$ the velocity v_r by $v_r^{(i)}$ and taking into account that the averaged admixture of v_L should not be relatively strong, one obtains as a rough estimate

$$\langle \varepsilon_r^2 \rangle / \bar{\varepsilon}^2 \sim (L/r)^{\mu} \sim v_L v_r^5 / v_r^6 \sim (L/r)^{1/3} \tag{14}$$

i.e., $\mu \approx 1/3$. The variance constant μ has been measured by many authors[3–7] and its modern values range within the interval $\mu \approx 0.2$–0.3. On the other hand, using the experimental data,[17,18] one can obtain [see also Eqs. (22a) and (22b) below] that $\alpha_1 \approx 0.5$ and $\alpha_2 \approx 0.7$ (with accuracy $\sim 20\,\%$). All these values are rather close to our rough estimates.

Substituting $\alpha_1 = \alpha_2 = 2/3$, $\mu = 1/3$ into inequality (8), one obtains the deviations from the unbounded log-normal distribution beginning at $|q| \gtrsim 3$, which is also in a good agreement with the experiments.

A partial improvement of these estimates can be obtained by subsequent iterations of the initial Kolmogorov values. This means that $\varepsilon_r^{1/3}$ and $\varepsilon_r^{2/3}$ in Eqs. (10) and (11), respectively, should be treated as $\langle \varepsilon_r^{1/3} \rangle$ and $\langle \varepsilon_r^{2/3} \rangle$, or

$$\alpha_1 \approx 2/3 + \mu_{1/3}$$

$$\alpha_2 \approx 2/3 - \mu_{2/3}$$

The corrections to μ in Eq. (14) are much more ambiguous, since $\langle \varepsilon_r^2 \rangle$ could be understood both in the sense $\langle \varepsilon_r^2 \rangle \sim v_L \langle v_r^5 \rangle / r^2$ and $\langle \varepsilon_r^2 \rangle \sim \bar{\varepsilon} v_L \langle v_r^2 \rangle / r$. Intuition based on the general theory of random processes (cf., e.g., the variations in a random addition) gives some preference to the second expression. Taking this form, one obtains the estimate

$$\mu_2 \equiv \mu \approx 1/3 + \mu_{2/3}$$

Since for low values of q, Eq. (5) gives a reasonably good approximation for $\mu_{1/3}$ and $\mu_{2/3}$, the corrected values of the exponents can easily be obtained in explicit form and are equal to $\mu \approx 0.3$, $\alpha_1 \approx 0.63$, and $\alpha_2 \approx 0.7$. Despite the heuristic character of these estimates, they give interesting information and may shed additional light on the problem of intermittency. In the general situation all these parameters should be treated as fitting ones.

In a recent paper, Meneveau and Sreenivasan[31] suggested the asymptotic value $D_\infty = 0.12 \pm 0.08$ (or $\alpha_1 = 0.88 \pm 0.08$) using the extreme tails of the probability distribution function $P(\varepsilon_r/\varepsilon_L)$. A reliable extrapolation to this extreme range is a very subtle procedure, which may, e.g., depend on the scaling resolution (on the box sizes of a net) and other factors. In these circumstance we should clearly state that although the bounded log-normal distribution extends the range of applicability of the log-normal approximation to higher orders of moments, nevertheless there exists some limiting value $q_{\mathrm{cr,max}}^{(\mathrm{bounded})}$ in the bounded case as well [cf. Eq. (8)]. The value $q_{\mathrm{cr,max}}^{(\mathrm{bounded})}$ is related to the fine structure of the probability distribution near the boundaries $\varepsilon_{\mathrm{min},r}$ and $\varepsilon_{\mathrm{max},r}$, since these values are also fluctuating. We have assumed the simplest possibility of the very abrupt decrease of $P(\varepsilon_r)$ beyond the interval $(\varepsilon_{\mathrm{min},r}, \varepsilon_{\mathrm{max},r})$ (much faster than the log-normal tails[31]). Thus, the infinite limits of the generalized dimensions D_q [see Eq. (21) below] in the bounded log-normal model should be understood in the sense of relatively high [with respect to interval (8)] values of $q \approx 10$–15 (or that obtained by the linear extrapolation from this range) rather than real infinity. The analytical criterion for $q_{\mathrm{cr,max}}^{(\mathrm{bounded})}$ depends on the fine structure of tails near $\varepsilon_{\min,r}$ and $\varepsilon_{\max,r}$ and cannot be written explicitly within the framework of our semiquantitative theory. The estimates for α_1 and α_2 obtained from D_q in the range of $|q| \approx$ 10–15 in ref. 31 give values all within the interval 0.5–0.7.

We end this section by describing some particular consequences of the application of the bounded log-normal distribution:

$$P(\varepsilon_r) = A_r \exp\left\{-\frac{[\ln(\varepsilon_r/\bar{\varepsilon}) + b_r\sigma_r^2]^2}{2\sigma_r^2}\right\} \tag{15}$$

$$\langle \varepsilon_r^q \rangle = \int_{\varepsilon_{\min,r}}^{\varepsilon_{\max,r}} d\varepsilon_r\, \varepsilon_r^q P(\varepsilon_r) \tag{16}$$

$$\langle \varepsilon_r^0 \rangle = 1, \qquad \langle \varepsilon_r \rangle = \bar{\varepsilon} \tag{17}$$

$$\sigma_r^2 \approx \mu_r \ln(L/r) \tag{18}$$

In the bounded log-normal model the parameters μ_r and b_r are weakly dependent functions of ln $\ln(L/r)$, since the scaling corrections should rigorously be absent in Eqs. (17). If the constants α_1 and α_2 are already fitted [see Eqs. (22) below], then a one-parameter family of functions μ_r and b_r is unambiguously fixed by any additional external restriction. The most usual condition is the normalization to the given experimental value of correlation exponent μ_2. Thus, in the most general case the bounded log-normal distribution is characterized by three fitting parameters α_1, α_2, and μ_2. However, since in the example concerned the variations in μ_r are very weak, we will give the direct value of μ_r.

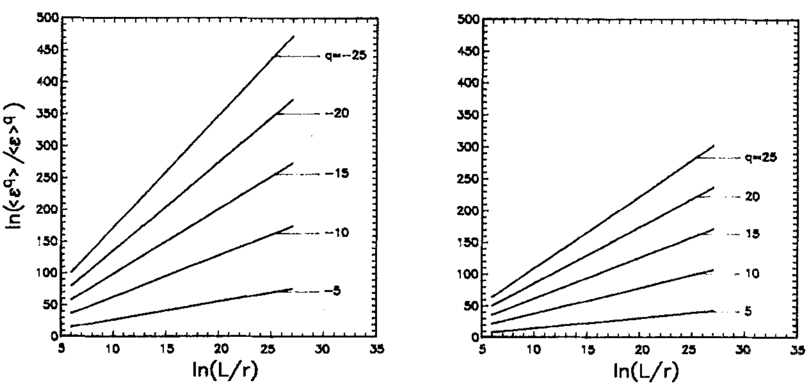

Fig. 2. Scaling behavior of moments $\langle \varepsilon_r^q \rangle$ obtained with the use of the bounded log-normal distribution (6), (7), and (15)–(18) with $\alpha_1 = 0.5$, $\alpha_2 = 0.7$, and $\mu_r = 0.245$.

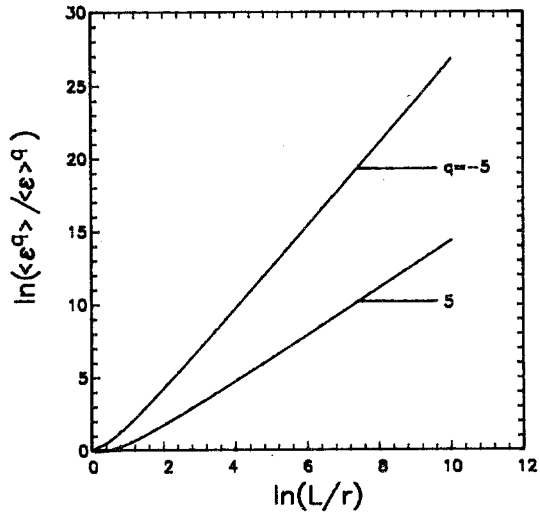

Fig. 3. The typical asymptotic outcomes to the scaling behavior for the averages $\langle \varepsilon_r^5 \rangle$ and $\langle \varepsilon_r^{-5} \rangle$ in the case of the bounded log-normal distribution (6), (7), and (15)–(18) with $\alpha_1 = 0.5$, $\alpha_2 = 0.7$, and $\mu_r = 0.245$.

We will utilize the experimental values by Meneveau and Sreenivasan,[17,18] $\alpha_1 = 0.5$, $\alpha_2 = 0.7$, and $\mu_2 = 0.25$. The parameter μ in Eq. (18) is determined by the condition

$$\langle \varepsilon_r^2 \rangle \propto r^{-\mu_2} \tag{19}$$

which gives $\mu_r = 0.245$ and $b_r = 1.5$ for $\mu_2 = 0.25$ (cf. the corresponding values $\mu = 0.25$ and $b = 3/2$ for the unbounded log-normal model). Figures 2 and 3 show the scaling behavior of the moments $\langle \varepsilon_r^q \rangle$. The values of μ_r and b_r have changed no more than 0.001 within the logarithmic range of $\ln(L/r)$ from 5 to 25. Figures 4 and 5 illustrate the dependences on q of constants C_q and μ_q defined according to

$$\langle \varepsilon_r^q \rangle = C_q \bar{\varepsilon}^q (L/r)^{\mu_q} \tag{20}$$

As is seen from Fig. 5, μ_q depends linearly on q in the limit of large $|q|$ in the case of the bounded log-normal distribution (solid line) in comparison with the quadratic dependence on $|q|$ for the unbounded distribution (dashed line), while for small q, both distributions coincide.

3. TWO-SCALE CANTOR SET MODEL

The characteristics of random processes can also be described with the use of multifractal theory[32] (see also the review in ref. 15). The

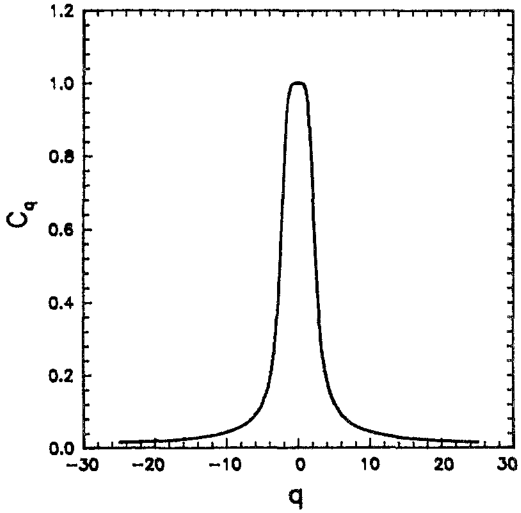

Fig. 4. The plot of C_q [see Eq. (20)] versus q for the bounded log-normal distribution with $\alpha_1 = 0.5$, $\alpha_2 = 0.7$, and $\mu_r = 0.245$.

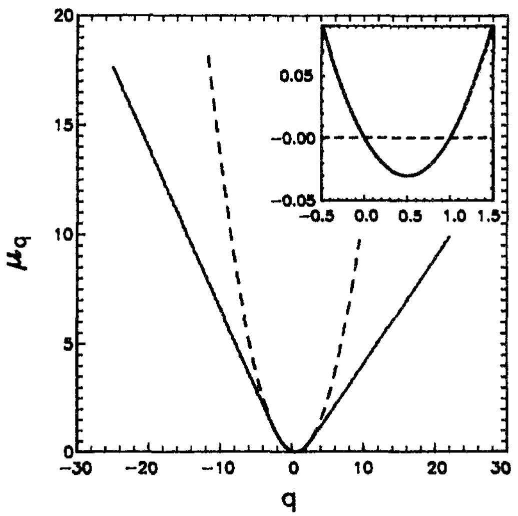

Fig. 5. Plots of μ_q [see Eq. (20)] versus q for the bounded log-normal distribution with $\alpha_1 = 0.5$, $\alpha_2 = 0.7$, and $\mu_r = 0.245$ (solid line) and for the unbounded log-normal distribution with $\mu = 0.25$ in Eq. (5) (dashed line).

correspondence of the multifractal description and the exponents μ_q in Eq. (20) is determined by the relationship

$$\mu_q = (D - D_q)(q - 1) \tag{21}$$

where D is the effective spatial dimension in a physical experiment, and $\{D_q\}$ are the so-called generalized Renyi dimensions. Comparing Eqs. (6), (7), (15)–(18), and (21), it is easy, in particular, to see that

$$\alpha_1 = D - D_\infty \tag{22a}$$

$$\alpha_2 = D_{-\infty} - D \tag{22b}$$

Meneveau and Sreenivasan[17,18,31] have studied the one-dimensional cuts of a turbulent flow ($D = 1$). They showed empirically that all of the experimental data for a variety of situations can be fitted by two-scale Cantor set dimensions,

$$D_q = \frac{1}{1-q} \log_2(0.7^q + 0.3^q) \tag{23}$$

within $\sim 10\%$ accuracy. For example, the experimental value of μ_2 is equal to 0.25 ± 0.05, while Eq. (23) gives $\mu_2 = 0.214$. Physically, the approximation (23) implies an unequal redistribution of energy during the consecutive process of eddy fragmentation, namely, it is supposed that one-half of the daughter eddies take fraction $p_1 = 0.7$ of energy of a mother eddy at each step of fragmentation, while the second half of the produced eddies take fraction $p_2 = 1 - p_1$ of an initial energy.[18]

Figure 6 ilustrates the distinction between the two-scale Cantor set approximation (23) (dotted line) from the results obtained with the use of the bounded log-normal distribution (solid line). The parameters α_1 , α_2, and μ_r [see Eqs. (6), (7), (15)–(18), and (22)] have been determined by the coincidence of D_∞, $D_{-\infty}$, and μ_2 with the corresponding values in Eq. (23). As is evident from Fig. 6, the two curves are practically indiscernible from the experimental point of view.

4. β-MODEL

Using in part the ideas of refs. 23 and 33, Frisch *et al.*[12] suggested the so-called homogeneous β-model of intermittency. It is supposed in this model that the centers of Kolmogorov eddies with sizes $\sim r$ ($L > r > l_\eta$) fill only a fraction of the physical space during fragmentation and energy transfer. A manifold of Kolmogorov eddies forms a homogeneous fractal

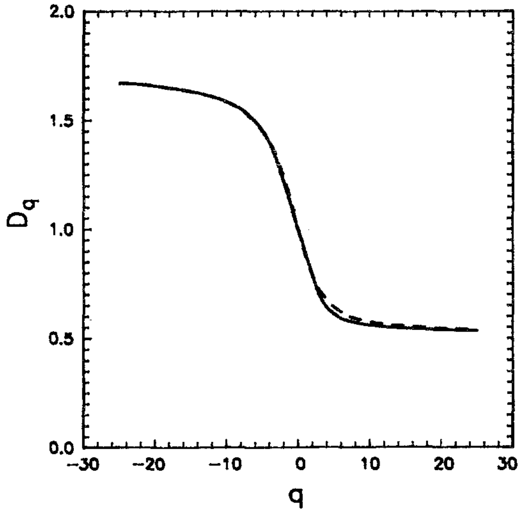

Fig. 6. The generalized Renyi dimension D_q versus q for the two-scale Cantor set [Eq. (23) and dashed line] and for the bounded log-normal distribution with $\alpha_1 = 0.485$, $\alpha^2 = 0.737$, and $\mu_r = 0.211$ [Eqs. (6), (7), (15)–(18), (21), and solid line].

with Hausdorff dimension D_f $(D > D_f)$. The corresponding rate of energy transfer is in this case given by

$$P_f(r)\, v_f^3(r)/r \sim \bar{\varepsilon} \tag{24}$$

where $P_f(r)$ is the probability that an arbitrarily chosen point of a fluid volume with sizes $\sim r$ belongs to the eddy fractal, and the subscript in $v_f(r)$ means that a turbulent velocity is taken on a fractal (these are the exceptional regions where the values of turbulent velocities are not equal to zero). In the case of a homogeneous fractal with dimension D_f the probability $P_f(r)$ has the form

$$P_f(r) \sim (r/L)^{D - D_f} \tag{25}$$

The various moments are determined according to

$$\langle \varepsilon_r^q \rangle \sim P_f(r)[v_f^3(r)/r]^q \sim \bar{\varepsilon}^q P_f^{1-q}(r) \sim \bar{\varepsilon}^q (L/r)^{(D - D_f)(q-1)} \tag{26}$$

i.e., the exponents μ_q [see Eq. (20)] are in the β-model equal to

$$\mu_q = (D - D_f)(q - 1) \tag{27}$$

and in particular,[23,12]

$$\mu_2 = D - D_f \tag{28}$$

The predictions of the homogeneous β-model are distinctly ruled out by experiments (see, e.g., refs. 6, 17, 18, and 31). For this reason in refs. 13 and 14 a multifractal generalization of the homogeneous β-model has been suggested. In this theory (called the random β-model) it is assumed that the curdling factors in the eddy cascade are mutually independent random variables at each step of consecutive fragmentation. For cascades with a contraction of scales by a factor of two, the random β-model gives for the exponents μ_q [13–15]

$$\mu_q = \log_2 \langle \beta^{1-q} \rangle \tag{29}$$

$$\langle \beta^{1-q} \rangle = \int_{\beta_{\min}}^{\beta_{\max}} d\beta\, P(\beta) \beta^{1-q} \tag{30}$$

$$\int_{\beta_{\min}}^{\beta_{\max}} d\beta\, P(\beta) = 1 \tag{31}$$

$$1 \geqslant \beta_{\max} > \beta_{\min} \geqslant 2^{-D} \tag{32}$$

where the probability $P(\beta)$ describes the distribution of the random space curdling factors β. Using Eqs. (21) and (29)–(32), one obtains that in the random β-model the following restrictions should be fulfilled:

$$\beta_{\min} = 2^{-(D - D_\infty)} \tag{33}$$

$$D \geqslant D_{-\infty} \tag{34}$$

Siebesma *et al.*[16] considered the random β-model with a partial correlation between various steps of fragmentation. It is, however, important that the restrictions (33) and (34) hold in their version as well.

5. COMPARISON OF VARIOUS MODELS. CONCLUSION

In order to compare the results of various models, we will again use the experimental data by Meneveau and Sreenivasan.[17,18] Following the original paper,[14] we restrict ourselves to the simple two-parameter approximation of the random β-model with the probability distribution $P(\beta)$ taken in the form

$$P(\beta) = (1-p)\, \delta(\beta - \beta_0) + p\, \delta(\beta - 1) \tag{35}$$

$$2^{-D} < \beta_0 < 1, \qquad 0 < p < 1 \tag{36}$$

Equation (35) means that the centers of Kolmogorov eddies fill the overall physical space with probability p, while with a probability $(1-p)$ they belong to a homogeneous fractal with Hausdorff dimension $DC + \log_2 \beta_0$. It follows from Eq. (33) that

$$\beta_0 = 2^{-(D - D_\infty)} \tag{37}$$

For this reason the fitting parameters β_0 and p can be chosen from the coincidence of D_∞ with that in Eq. (23) and the condition $\mu_2 = 0.25$. This gives $p = 0.53$, $\beta_0 = 2^{-0.5} \approx 0.71$, and

$$\mu_q = (q-1)(D - D_q) = \log_2[0.47 \times 2^{-0.5(1-q)} + 0.53] \tag{38}$$

The corresponding results for the generalized Renyi dimensions in the various models are summarized in Fig. 7. Choosing a more flexible fit for a probability $P(\beta)$, the correspondence for D_q in the random β-model can further be improved to the range of positive $q > 0$. However, the principal inequality (34), $D \geqslant D_{-\infty} \geqslant D_q$, breaks the possible correspondence to the range of negative $q < 0$, if $D_q > D$ at $q < 0$ (see also ref. 31). Since the excess of D_q over D at $q < 0$ has been observed in refs. 17, 18, and 31 well within the experimental errors, these data can satisfactorily be described only in the framework of the two-scale Cantor set and bounded log-normal models. It seems that at the present state of art the theoretical background

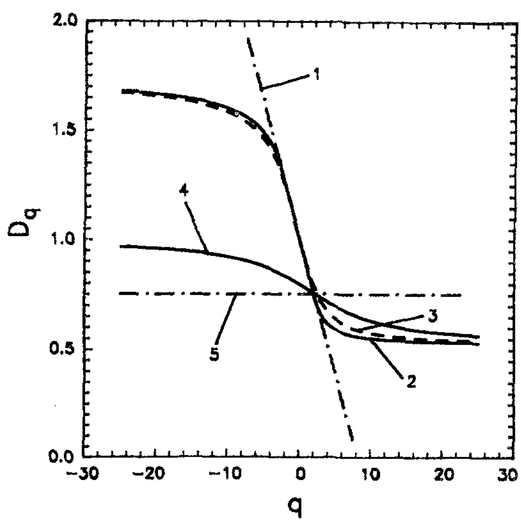

Fig. 7. The generalized Renyi dimension in the various models of intermittency. (1) Unbounded log-normal distribution with $\mu = 0.25$; (2) bounded log-normal distribution with $\alpha_1 = 0.5$, $\alpha_2 = 0.7$, and $\mu_r = 0.245$ (solid line); (3) two-scale Cantor set model [Eq. (23) and dashed line]; (4) the random β-model [Eq. (38)]; (5) the homogeneous β-model with $\mu_2 = 0.25$ [Eq. (27)].

of the bounded log-normal model looks more sound. Thus, the application of the bounded log-normal model resolves some difficulties related to the original unbounded Kolmogorov–Obukhov distribution and appears to be in good agreement with the experimental data.

As is well known,[23] the log-normal distribution gives practically universal description for low values of q (or near maximum in f–α curves[18,31,34]). On the other hand, the high-order moments $\langle \varepsilon_r^q \rangle$ (or the wings of f–α curves) are determined by the rare events related to the characteristic maximum and minimum level of fluctuations (see for discussion, e.g., refs. 31 and 34 and further references therein). For this reason the bounded log-normal distribution may give a rather universal and adequate description of the intermittency throughout a wide range of the parameters.

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